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Entanglement and entropy rates in open quantum systems

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Abstract
We study a recent conjecture about the behavior of the quantum relative entropy compared to the relative entropy of entanglement in open bipartite systems. The conjecture states that, under a dissipative time evolution, the positive rate of change of the relative entropy will always be larger than that of the relative entropy of entanglement. After explicitly solving a 2-qubit master equation of Lindblad type with separable and entangled stationary states, we show that the conjecture can be violated for initial states with an entangled asymptotic state, while it appears to be confirmed when the asymptotic states are separable.

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1. Introduction

The importance of quantum entanglement as a physical resource for performing informational tasks which would be classically impossible [1] has spurred the study of its dynamical behavior in many different systems. The time evolution of most of these is reversible and generated by a Hamiltonian; however, it is important for quantum entanglement to be used as an efficient physical resource so that its temporal behavior can also be studied when systems are driven by noisy environments and their dynamics is irreversible.

In the following, we will consider *open quantum systems* [2–4], i.e. systems where the interactions between the subsystem S and the external environment E, though weak, cannot be neglected. In this case, a standard way of obtaining a manageable dissipative time evolution of the density matrix ϱ_t describing the state of S at time t is to construct it as the solution of a Liouville-type master equation: $\partial_t \varrho_t = \mathbf{L}[\varrho_t]$, where the generator $\mathbf{L}$ of Lindblad type [5, 6] takes care of the effects of the environment through a characteristic matrix of coefficients known as *Kossakowski matrix*. This can be done by tracing away the environment degrees of freedom and by performing a Markovian approximation, i.e. by studying the evolution on a slow time scale and neglecting fast decaying memory effects. Then, the irreversible reduced dynamics of S is described by one-parameter semigroups of linear maps, called *quantum dynamical semigroups*, obtained by exponentiation: $\gamma_t = \mathrm{e}^{t\mathbf{L}}$, $t \geqslant 0$, such that $\varrho_t \equiv \gamma_t[\varrho]$. In order to guarantee full physical consistency, namely that $\mathrm{id} \otimes \gamma_t$ be positivity preserving on all states of the compound system $S + S_d$ for any inert ancilla S_d, γ_t must be completely positive [2, 3, 7].

The formalism of open quantum systems has been used to describe the tendency to thermal equilibrium of a small system in weak interaction with a large heat bath at a certain temperature. The main tool in this thermodynamical picture is the quantum relative entropy [3]; it is related to the difference between the free energy of the irreversibly evolving open quantum system and that of its equilibrium asymptotic state: this difference monotonically decreases in time because so does the quantum relative entropy with respect to completely positive maps [8], as quantum dynamical semigroups are. Namely, the time derivative of the quantum relative entropy, called *entropy rate*, has a definite sign.

The quantum relative entropy has also been used as a possible measure of the entanglement content of a quantum state; the so-called *relative entropy of entanglement* provides a pseudo-distance between a state and the closed convex set of separable states [9].

In [10] the natural question was raised whether the entropy production due to thermodynamical tendency to equilibrium is somewhat related to the *entanglement rate*, that is to the speed of variation of the relative entropy of entanglement. A conjecture was put forward that for systems immersed in an external bath without a direct source of entanglement due to Hamiltonian interactions, the absolute value of the entanglement production is always smaller than the entropy production.

Typically, a system S immersed in a large environment E is subjected to decoherence; therefore, one expects quantum entanglement to be generically depleted by a dissipative and noisy time evolution. The conjecture mentioned above is motivated by the fact that, if a quantum open system tends to a separable equilibrium state, then, in a suitable neighborhood of the latter, the entanglement production is zero while the entropy production is not. Indeed, in [10] a concrete example that validates the conjecture is offered of a 2-qubit system in which only one of them evolves as a quantum open system. In such a case an initial maximally entangled state evolves toward a separable steady state with an entropy production always larger than the speed with which entanglement is dissipated.

However, in certain specific situations, an environment affecting both parties of a bipartite system may even build quantum correlations between the subsystems which compose S (see, e.g. [7, 11–14]). In particular, in [7] this possibility is shown to depend on the specific form of the generator of the reduced dynamics. In [15] an inequality was found, involving the entries of such a matrix which, if fulfilled, is sufficient to ensure that a specific initial separable pure state of two qubits gets entangled. Further, in [16] this inequality was proven to be a necessary and sufficient condition for environment-induced entanglement in an initially separable pure state of two qubits. Even more interestingly, starting from an initially separable state, the entanglement generated at small times can persist asymptotically; also, starting from an initially entangled state, its entanglement content can asymptotically increase.

This work is organized as follows: in section 2, we consider the open dynamics of two qubits with a generator that depends on a parameter which allows to range over all the above-mentioned cases, analytically solving the master equation; then, in section 3, we introduce the notions of entropy and entanglement rates and the conjecture from [10]; finally, in section 4, numerically studying the time behavior of the entanglement and entropy rates for various initial states, we show that, whenever there is asymptotic entanglement the conjecture in [10] is violated, while it holds if there is no asymptotic entanglement.

2. The reduced dynamics

Let a bipartite system composed of two qubits be immersed in an external environment in such a way that, via standard weak-coupling limit techniques [2], one describes their reduced, irreversible dynamics by means of the master equation

$$\partial_t \varrho_t = \mathbf{L}[\varrho_t] = -\mathrm{i}\,\frac{\Omega}{2}[\Sigma_3, \varrho_t] + \sum_{i,j=1}^{3} A_{ij}\left(\Sigma_i \varrho_t \Sigma_j - \frac{1}{2}\{\Sigma_j \Sigma_i, \varrho_t\}\right), \tag{1}$$

where Ω is the system frequency, $\Sigma_i := \sigma_i \otimes \mathbf{I} + \mathbf{I} \otimes \sigma_i$, $\mathbf{I}$ is the 2×2 identity matrix, σ_i, $i = 1, 2, 3,$ are the Pauli matrices and the matrix

$$A = [A_{ij}] = \begin{pmatrix} 1 & \mathrm{i}\alpha & 0 \\ -\mathrm{i}\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \alpha \in \mathbb{R}, \quad \alpha^2 \leqslant 1, \tag{2}$$

is positive semi-definite. This latter request ensures that the semigroup generated by (1) consists of completely positive maps γ_t for all $t \geqslant 0$ [2].

Remark 1. By means of the single-qubit Pauli matrices $\sigma_i^{(1)} = \sigma_i \otimes \mathbf{I}$ and $\sigma_i^{(2)} = \mathbf{I} \otimes \sigma_i$, one writes the purely dissipative contribution to the generator as [7]

$$\mathbf{D}[\rho_t] = \sum_{i,j=1}^{3} A_{ij} \sum_{a,b=1}^{2} \left(\sigma_i^{(a)}\, \rho_t\, \sigma_j^{(b)} - \frac{1}{2}\left\{\sigma_j^{(b)} \sigma_i^{(a)}, \rho_t\right\}\right). \tag{3}$$

In this way there are six Kraus operators $\sigma_i^{(a)}$, $a = 1, 2$, $i = 1, 2, 3$, and the 6×6 Kossakowski matrix reads

$$K = \left[K_{ij}^{(ab)}\right] = \begin{pmatrix} K^{(11)} & K^{(12)} \\ K^{(21)} & K^{(22)} \end{pmatrix} = \begin{pmatrix} A & A \\ A & A \end{pmatrix}. \tag{4}$$

From the theory of open quantum systems [2, 5, 3, 4] one knows that the coefficients $K_{ij}^{(ab)}$ in the Kossakowski matrix relative to the ith Pauli matrix of the ath qubit, respectively the jth Pauli matrix of the bth qubit, $a, b = 1, 2$, $i, j = 1, 2, 3$, are determined by the Fourier transforms of the two-point time-correlation functions with respect to an environment equilibrium state ω, $\omega\big(B_i^{(a)} B_j^{(b)}(t)\big)$, of the environment operators $B_i^{(a)}$ appearing in the system–environment interaction $H_I = \sum_{i=1}^{3}\big(\sigma_i^{(1)} \otimes B_i^{(1)} + \sigma_i^{(2)} \otimes B_i^{(2)}\big)$. The symmetric form of (3) thus results when both qubits are linearly coupled to bath operators such that: $B_{1,2,3}^{(1)} = B_{1,2,3}^{(2)} = B_{1,2,3}$ and $\omega(B_{1,2} B_3(t)) = 0$.

Remark 2. Considering two qubits weakly interacting with a thermal bath modeled as a collection of spinless, massless scalar fields (see, e.g. [7]) at very high temperature $T = 1/\beta$, the parameter α in the Kossakowski matrix is related to β, i.e. $\alpha = -\beta\Omega$, where Ω is the system frequency when isolated from the environment. Correspondingly, in the case of one qubit immersed in such a thermal bath at high temperature ($\beta \ll 1$), any initial state is driven to the thermal asymptotic state

$$\rho_\infty = \frac{\exp(-\beta\Omega\sigma_3)}{2\cosh\beta\Omega} \simeq \frac{1}{2}(1 - \beta\Omega\sigma_3).$$

The master equation (1) is explicitly integrated in appendix A; in the following we will mainly focus upon the time evolution of initial states of the form

$$\varrho = a\,|1\rangle\langle 1| + d\,|2\rangle\langle 2| + b\,|3\rangle\langle 3| + c\,|4\rangle\langle 4|, \qquad a, b, c, d \in \mathbb{R}^+, \qquad a + b + c + d = 1, \tag{5}$$

diagonal with respect to the orthonormal vectors

$$|1\rangle = |00\rangle, \qquad |2\rangle = |11\rangle, \qquad |3\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \qquad |4\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}, \tag{6}$$

where $\sigma_3|0\rangle = |0\rangle$, $\sigma_3|1\rangle = -|1\rangle$ and $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ form the so-called standard basis in $\mathbb{C}^2 \otimes \mathbb{C}^2$, with respect to which the states (5) are represented by

$$\varrho = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \frac{b+c}{2} & \frac{b-c}{2} & 0 \\ 0 & \frac{b-c}{2} & \frac{b+c}{2} & 0 \\ 0 & 0 & 0 & d \end{pmatrix}. \tag{7}$$

From equations (A.4)–(A.9) and (A.11) in appendix A, it turns out that these initial states evolve at time $t \geqslant 0$ into states of the same form

$$\varrho_t = a_t\,|1\rangle\langle 1| + d_t\,|2\rangle\langle 2| + b_t\,|3\rangle\langle 3| + c_t\,|4\rangle\langle 4|, \tag{8}$$

where $c_t = c$ and

$$a_t = \frac{(1-\alpha)^2}{3+\alpha^2} R + \sqrt{1-\alpha^2}\,\frac{(1+\alpha)^2 a\; - 2(1-\alpha)\,d + (1+\alpha)^2\,b}{(1+\alpha)(3+\alpha^2)} E_-(t) + \frac{2(1+\alpha)\,a\; - (1-\alpha)^2(b+d)}{3+\alpha^2} E_+(t) \tag{9}$$

$$d_t = \frac{(1+\alpha)^2}{3+\alpha^2} R - \sqrt{1-\alpha^2}\,\frac{2(1+\alpha)\,a\; - (1-\alpha)^2(b+d)}{(1-\alpha)(3+\alpha^2)} E_-(t) - \frac{(1+\alpha)^2\,a\; - 2(1+\alpha)\,d\; + (1+\alpha)^2\,b}{3+\alpha^2} E_+(t) \tag{10}$$

$$b_t = \frac{(1-\alpha^2)}{3+\alpha^2} R + \sqrt{1-\alpha^2}\,\frac{(1+\alpha)^3\,a\; + (1-\alpha)^3\,d\; - 2(1-\alpha^2)\,b}{(3+\alpha^2)(1-\alpha^2)} E_-(t) + \frac{2(1+\alpha^2)\,b\; - (1-\alpha^2)(a+d)}{3+\alpha^2} E_+(t), \tag{11}$$

with $R = a + b + d = 1 - c$ and

$$E_+(t) = \mathrm{e}^{-8t}\cosh 4t\sqrt{1-\alpha^2}, \qquad E_-(t) = \mathrm{e}^{-8t}\sinh 4t\sqrt{1-\alpha^2}.$$

Since $\lim_{t\to+\infty} E_\pm(t) = 0$, the asymptotic states resulting from the initial states (5) are

$$\varrho_\infty(c) = \frac{(1-\alpha)^2}{3+\alpha^2}(1-c)\,|1\rangle\langle 1| + \frac{(1+\alpha)^2}{3+\alpha^2}(1-c)\,|2\rangle\langle 2| + \frac{(1-\alpha^2)}{3+\alpha^2}(1-c)\,|3\rangle\langle 3| + c\,|4\rangle\langle 4|. \tag{12}$$

There is thus a one-parameter family $\{\varrho_\infty(c)\}_{0\leqslant c\leqslant 1}$ of asymptotic states such that all initial states of the form (5) with the same c go into the same $\varrho_\infty(c)$.

In order to study the asymptotic entanglement generation capability of the present model, we shall measure the entanglement of 2-qubit states ρ by the concurrence [19]

$$C(\varrho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant 0$ are the square roots of the positive eigenvalues of $\varrho\widetilde{\varrho}$ with $\widetilde{\varrho} = \sigma_2 \otimes \sigma_2\, \varrho^*\, \sigma_2 \otimes \sigma_2$, ϱ^* denoting the complex conjugated matrix. For 2-qubit states of the form (7), $\rho = \rho^*$ and one easily computes

$$\widetilde{\varrho} = \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & \frac{b+c}{2} & \frac{b-c}{2} & 0 \\ 0 & \frac{b-c}{2} & \frac{b+c}{2} & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \qquad \varrho\widetilde{\varrho} = \begin{pmatrix} ad & 0 & 0 & 0 \\ 0 & \frac{b^2+c^2}{2} & \frac{b^2-c^2}{2} & 0 \\ 0 & \frac{b^2-c^2}{2} & \frac{b^2+c^2}{2} & 0 \\ 0 & 0 & 0 & ad \end{pmatrix}.$$

This latter matrix has positive eigenvalues ad (twice degenerate), b^2, c^2; then, their square roots $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$ in decreasing order yield

$$C(\varrho) = \max\left\{0, 2\left(\frac{|b-c|}{2} - \sqrt{ad}\right)\right\} \tag{13}$$

$$C(\varrho_\infty) = \max\left\{0, \frac{|1-\alpha^2-4c| - 2(1-\alpha^2)(1-c)}{3+\alpha^2}\right\}. \tag{14}$$

Remark 3. As already emphasized in the introduction, despite decoherence, the presence of an environment need not have only destructive effects in relation to entanglement: entanglement can even be asymptotically increased with respect to the initial amount. This can happen in the present case and the entanglement generation capability of the environment is entirely due to the non-Hamiltonian contribution (3) to the generator in (1). Indeed, the 2-qubit Hamiltonian does not contain coupling terms and cannot be a source of entanglement; instead, this can be true for (3) because the off-diagonal contributions in the Kossakowski matrix (4) couple the two qubits. Of course, this is only necessary, but not sufficient to ensure entanglement generation and its asymptotic persistence. They indeed depend on a trade-off between the off-diagonal couplings and the purely decohering diagonal terms in (4).

3. Entropy and entanglement rates

In this section we shall introduce the notions of entropy and entanglement rates; for the sake of simplicity, we shall consider finite d-level systems whose states are described by normalized, positive, $d \times d$ density matrices $\varrho \in M_d(\mathbb{C})$. Given two such density matrices, their quantum relative entropy is defined by [8]

$$S(\rho_1 \| \rho_2) = \mathrm{Tr}(\varrho_1(\log \varrho_1 - \log \varrho_2)). \tag{15}$$

Consider a quantum system with Hamiltonian H; if in contact with a heat bath at temperature $T = 1/\beta$ (with the Boltzmann constant $\kappa = 1$), it is expected to be driven asymptotically into the thermal (Gibbs) state $\varrho_T = \mathrm{e}^{-\beta H}/Z_\beta$, where $Z_\beta = \mathrm{Tr}[\mathrm{e}^{-\beta H}]$. Suppose that under an irreversible time evolution $\varrho \mapsto \varrho_t$, an initial state ϱ is driven into thermal equilibrium, that is $\lim_{t\to+\infty} \varrho_t = \varrho_T$; then,

$$\frac{1}{\beta} S(\varrho_t \| \varrho_T) = \frac{1}{\beta} \mathrm{Tr}(\varrho_t(\log \varrho_t + \log Z_\beta + \beta H)) = -T S(\varrho_t) + \mathrm{Tr}(\varrho_t H) + T \log Z_\beta,$$

where $S(\varrho) = -\mathrm{Tr}\, \varrho \log \varrho$ is the von Neumann entropy of the state ρ. Since the second term corresponds to the system's internal energy, the first two contributions give the system's free energy corresponding to the time-evolving state ρ_t [3]:

$$F(\varrho_t) = U(\varrho_t) - T S(\varrho_t), \qquad U(\varrho_t) = \mathrm{Tr}(\varrho_t H).$$

Finally, $F(\varrho_T) = -\log Z_\beta$ implies that the quantum relative entropy is related to the difference of free energies

$$S(\varrho_t\|\varrho_T) = \beta(F(\varrho_t) - F(\varrho_T)).$$

Because of the second law of thermodynamics, the above quantity should be positive and its time derivative non-positive. The first property is guaranteed by the properties of the quantum relative entropy [8], while the second one holds true when the irreversible time evolution is given by a Markovian semigroup, that is when $\varrho_t = \gamma_t[\varrho]$ and $\gamma_t \circ \gamma_s = \gamma_s \circ \gamma_t = \gamma_{s+t}$ for all $s, t \geqslant 0$. Indeed, since $\gamma_t[\varrho_T] = \varrho_T$, one derives

$$\begin{aligned} S(\varrho_t\|\varrho_T) &= S(\gamma_t[\varrho]\|\gamma_t[\varrho_T]) = S(\gamma_{t-s}\circ\gamma_s[\varrho]\|\gamma_{t-s}\circ\gamma_s[\varrho_T]) \\ &\leqslant S(\gamma_s[\varrho]\|\gamma_s[\varrho_T]) = S(\varrho_s\|\varrho_T) \quad \forall\, 0 \leqslant s \leqslant t, \end{aligned}$$

where the last inequality follows from the fact that the quantum relative entropy decreases under the action of completely positive trace-preserving maps [8].

Based on the previous thermodynamical arguments, one may consider generic open quantum dynamics $\varrho \mapsto \gamma_t[\varrho] = \varrho_t$ with asymptotic states $\lim_{t\to+\infty} \varrho_t = \varrho_\infty$ that are not necessarily thermal ones. The speed of convergence to such stationary states starting from an initial state ϱ will then be measured by the *entropy rate*

$$\sigma[\varrho_t] = -\frac{\mathrm{d}}{\mathrm{d}t} S(\varrho_t\|\varrho_\infty) = \mathrm{Tr}(\dot{\varrho}_t(\log\varrho_\infty - \log\varrho_t)). \tag{16}$$

The entropy production that accompanies the tendency to equilibrium of the states of the form (8) is easily computed; indeed, being the states ϱ_t and ϱ_∞ diagonal with respect to the same orthonormal basis, the entropy rate (16) has the analytic expression

$$\sigma[\varrho_t] = \dot{a}_t \log\frac{(1-\alpha)^2(1-c)}{a_t(3+\alpha^2)} + \dot{b}_t \log\frac{(1-\alpha^2)(1-c)}{b_t(3+\alpha^2)} + \dot{d}_t \log\frac{(1+\alpha)^2(1-c)}{d_t(3+\alpha^2)}. \tag{17}$$

When the density matrix ρ is the state of, say, a bipartite quantum system, it makes sense to introduce the *relative entropy of entanglement*,

$$E[\varrho] = \inf_{\varrho_{\mathrm{sep}}} S(\varrho\|\varrho_{\mathrm{sep}}), \tag{18}$$

as a measure of the entanglement content of ρ. Indeed, the above quantity vanishes if and only if ρ is separable and can be used to measure the distance of ρ [4] from the convex set of separable states; furthermore, it cannot increase, but at most remains constant, under the action of local operations, described by trace-preserving completely positive maps acting independently on the two parties [17, 18].

Analogously to what was done for the entropy production, one may look at the *entanglement rate* when the system evolves, i.e. at the time derivative of the pseudo-distance

$$\sigma_E[\varrho_t] = \frac{\mathrm{d}}{\mathrm{d}t} E[\varrho_t]. \tag{19}$$

In [10] it was argued that

$$|\sigma_E[\varrho_t]| \leqslant \sigma[\varrho_t] \tag{20}$$

always holds in the absence of direct entangling interactions between the parties. The argument on which the conjecture is based is that decoherence is expected to deplete entanglement before reaching the asymptotic state and thus before the entropy production vanishes. Such asymptotic intuition is then extrapolated at all times.

[4] The relative entropy of entanglement is not exactly a distance since it is not symmetric.

Now we will illustrate the various possibilities offered by the reduced dynamics discussed in the previous section; in particular, we will compare the entropy and entanglement rates, thus checking the validity of the conjecture (20).

We will first derive an explicit expression for the relative entropy of entanglement (18) in the case of states of the form (8) and then compute numerically the behavior of its time derivative (19).

Let us first rewrite (18) as follows:

$$E(\varrho_t) = -S(\varrho_t) - \sup_{\varrho_{\rm sep}} \mathrm{Tr}(\varrho_t \log \varrho_{\rm sep}), \qquad (21)$$

where $S(\varrho_t)$ is the von Neumann entropy of the time-evolving state. The following result, which we prove in appendix B, helps to explicitly solve the above maximization problem.

Proposition 1. *In the case of states as in (8), the supremum in (21) is achieved for separable states of the form*

$$\varrho_{\rm sep} = x|1\rangle\langle 1| + u|3\rangle\langle 3| + v|4\rangle\langle 4| + y|2\rangle\langle 2|, \qquad (22)$$

where the parameters x, y, u, v are real and such that

$$x + u + v + y = 1, \qquad \frac{|u - v|}{2} \leqslant \sqrt{xy}. \qquad (23)$$

This leads to the following maximization problem

$$E(\varrho_t) = -S(\varrho_t) - \sup_{\varrho\in\mathcal{S}_{\rm sep}^{\rm diag}} (a_t \log x + d_t \log y + b_t \log u + c_t \log v), \qquad (24)$$

which can be analytically solved.

4. Results

The above maximization problem is explicitly solved in appendix B thus permitting to calculate numerically the entanglement rate (19), to compare it with the entropy rate (16) and to check the conjecture (20). We shall do this in a number of cases that cover all possible initial and asymptotic entanglement properties for which we plot the behaviors of the relative entropy and of the relative entropy of entanglement, separately, while the entropy rate and the entanglement rate are plotted together for direct comparison.

In the following, the choice of the range of values for the plots' axes was made only for graphic reasons to make the plots clearer. Moreover, in the first four cases, we took the parameter in the matrix A to be $\alpha = 0.5$ as it makes the plots easier to read; changing α does not alter the results. In the last example, instead, we show two different behaviors of the entanglement of the initial state depending on the choice of the parameter α.

Case 1. An initial pure separable state (25) goes into a mixed separable state; the dissipative time evolution is not able to generate entanglement at any time as shown in figures 1 (right) and 2, where the entropy of entanglement and the entanglement rate are both zero. In this case the conjecture (20) holds:

$$\begin{aligned} &\varrho = |1\rangle\langle 1| \\ &\varrho_\infty = \frac{(1-\alpha)^2}{3+\alpha^2}\,|1\rangle\langle 1| + \frac{(1+\alpha)^2}{3+\alpha^2}\,|2\rangle\langle 2| + \frac{(1-\alpha^2)}{3+\alpha^2}\,|3\rangle\langle 3| \\ &C(\varrho) = C(\varrho_\infty) = 0. \end{aligned} \qquad (25)$$

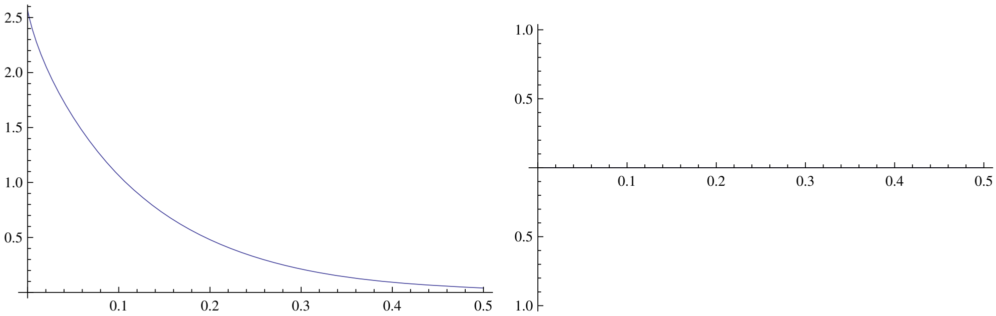

Figure 1. Case 1: $\alpha = 0.5$; left: $S(\varrho_t)$, right: $E(\varrho_t)$.

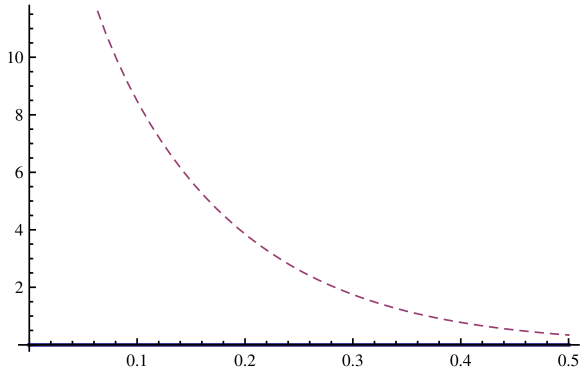

Figure 2. Case 1: $\alpha = 0.5$; $\sigma[\varrho_t]$ dashed line, $|\sigma_E[\varrho_t]|$ continuous line.

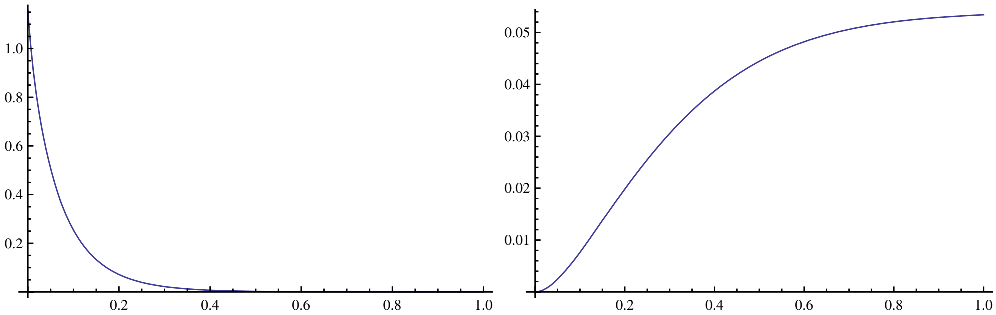

Figure 3. Case 2: $\alpha = 0.5$; left: $S(\varrho_t\|\varrho_\infty)$, right: $E[\varrho_t]$

Case 2. An initial mixed separable state (26) goes into a mixed entangled state and the conjecture (20) is violated after some time (see figures 3 and 4):

$$\varrho = \frac{1}{2}\,|3\rangle\langle 3| + \frac{1}{2}\,|4\rangle\langle 4|$$
$$\varrho_\infty = \frac{(1-\alpha)^2}{2(3+\alpha^2)}\,|1\rangle\langle 1| + \frac{(1+\alpha)^2}{2(3+\alpha^2)}\,|2\rangle\langle 2| + \frac{(1-\alpha^2)}{2(3+\alpha^2)}\,|3\rangle\langle 3| + \frac{1}{2}\,|4\rangle\langle 4| \qquad (26)$$
$$C(\varrho) = 0, \qquad C(\varrho_\infty) = \frac{2\alpha^2}{3+\alpha^2} \geqslant 0 \quad \forall \alpha \in [-1, 1].$$

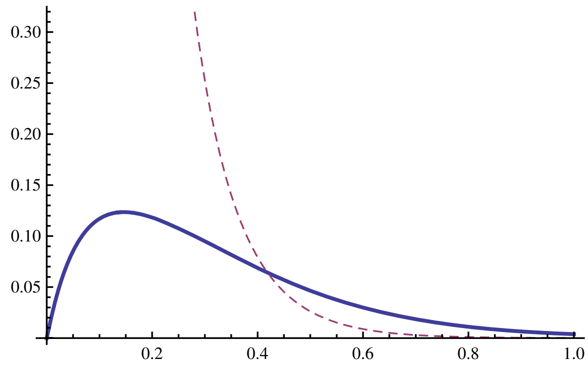

Figure 4. Case 2: $\alpha = 0.5$; $\sigma[\varrho_t]$ dashed line, $|\sigma_E[\varrho_t]|$ continuous line.

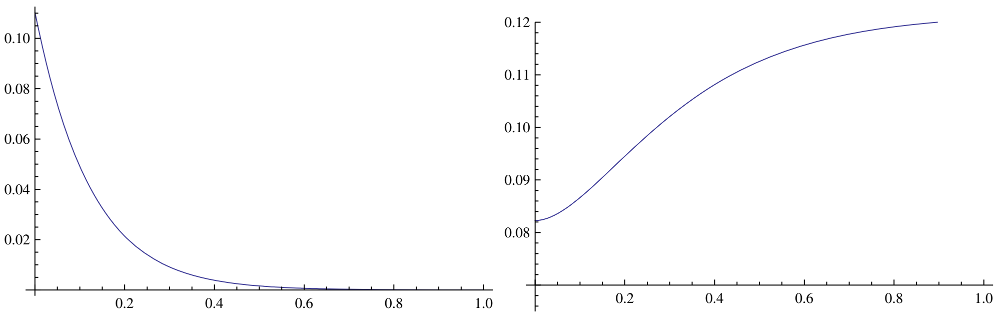

Figure 5. Case 3: $\alpha = 0.5$; left: $S(\varrho_t \| \varrho_\infty)$, right: $E[\varrho_t]$.

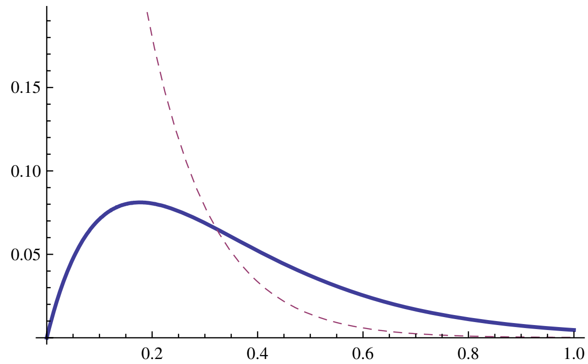

Figure 6. Case 3: $\alpha = 0.5$; $\sigma[\varrho_t]$ dashed line, $|\sigma_E[\varrho_t]|$ continuous line.

Case 3. An initial mixed entangled state (27) goes into an asymptotic mixed state which is more or equally entangled and the conjecture (20) is violated after some time (see figures 5 and 6):

$$\varrho = \frac{1}{10}\,|1\rangle\langle 1| + \frac{1}{10}\,|2\rangle\langle 2| + \frac{1}{10}\,|3\rangle\langle 3| + \frac{7}{10}\,|4\rangle\langle 4|$$

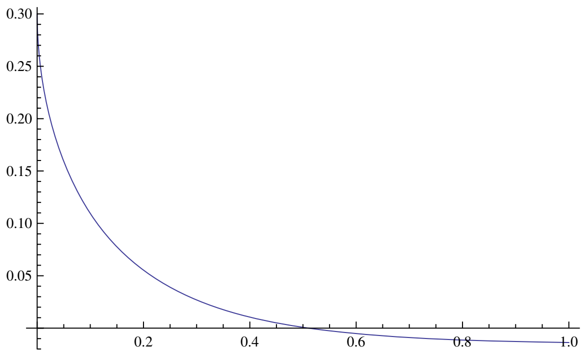

Figure 7. Case 4: $\alpha = 0.5$; $C(\varrho_t)$.

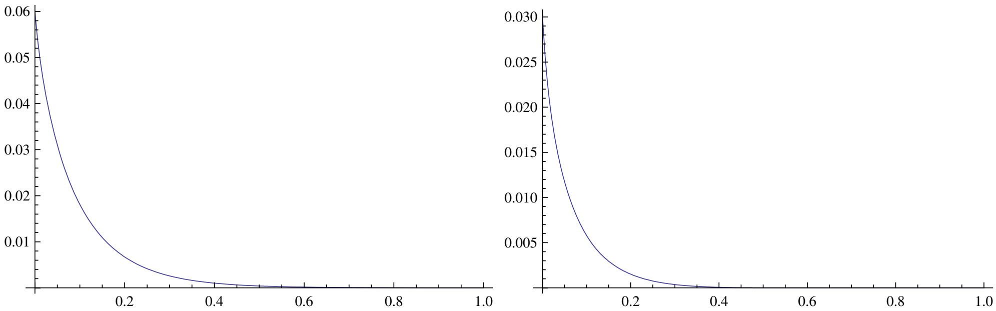

Figure 8. Case 4: $\alpha = 0.5$; left: $S(\varrho_t \| \varrho_\infty)$, right: $E[\varrho_t]$.

$$\varrho_\infty = \frac{3(1-\alpha)^2}{10(3+\alpha^2)}\,|1\rangle\langle 1| + \frac{3(1+\alpha)^2}{10(3+\alpha^2)}\,|2\rangle\langle 2| + \frac{3(1-\alpha^2)}{10(3+\alpha^2)}\,|3\rangle\langle 3| + \frac{7}{10}\,|4\rangle\langle 4|$$
$$C(\varrho) = \frac{2}{5}, \qquad C(\varrho_\infty) = \frac{2}{5}\frac{3+4\alpha^2}{3+\alpha^2} \geqslant \frac{2}{5} \quad \forall \alpha \in [-1, 1]. \tag{27}$$

Case 4. An initial mixed entangled state (28) goes into a state with less entanglement:

$$\varrho = \frac{1}{2}|2\rangle\langle 2| + \frac{1}{10}\,|3\rangle\langle 3| + \frac{2}{5}\,|4\rangle\langle 4|$$
$$\varrho_\infty = \frac{3(1-\alpha)^2}{5(3+\alpha^2)}\,|1\rangle\langle 1| + \frac{3(1+\alpha)^2}{5(3+\alpha^2)}\,|2\rangle\langle 2| + \frac{3(1-\alpha^2)}{5(3+\alpha^2)}\,|3\rangle\langle 3| + \frac{2}{5}\,|4\rangle\langle 4| \tag{28}$$
$$C(\varrho) = \frac{3}{5}, \qquad \begin{cases} C(\varrho_\infty) = 0 & \text{for} \quad \alpha^2 \leqslant \frac{3}{11} \\ C(\varrho_\infty) = \frac{11\alpha^2-3}{5(3+\alpha^2)} < \frac{3}{5} & \text{for} \quad \frac{3}{11} < \alpha^2 \leqslant 1. \end{cases}$$

With the choice $\alpha = 0.5$, the dissipative time evolution shows a sudden death of entanglement, that is, the concurrence[5] (13) vanishes at finite time. The conjecture (20) always holds (see figures 7, 8 and 9).

[5] In figure 7, instead of plotting the concurrence as defined in (13), we have plotted the difference $|b-c| - 2\sqrt{ad}$; this simply means that, as soon as this difference becomes negative, the state is separable.

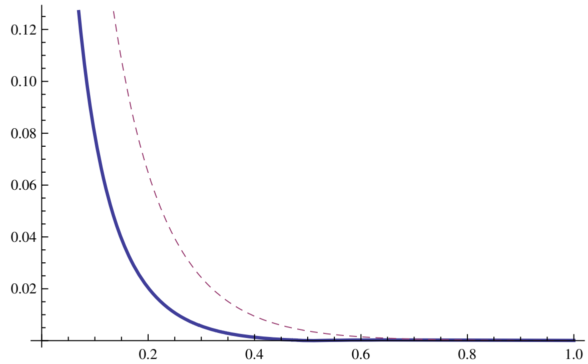

Figure 9. Case 4: $\alpha = 0.5$; $\sigma[\varrho_t]$ dashed line, $|\sigma_E[\varrho_t]|$ continuous line.

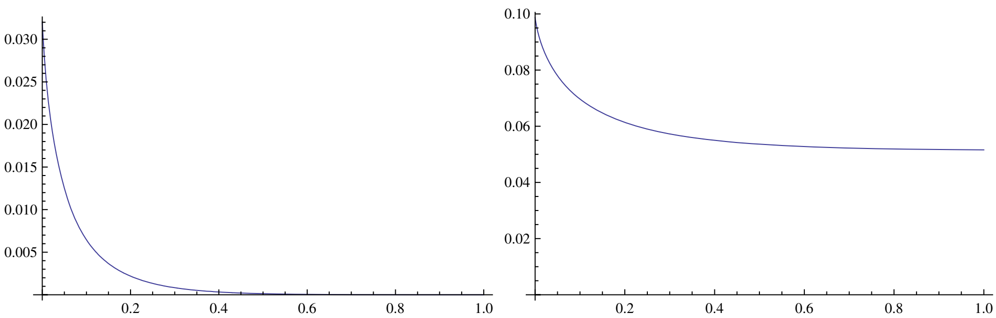

Figure 10. Case 5: $\alpha = 0.5$; left: $S(\varrho_t\|\varrho_t)$, right: $E[\varrho_t]$.

Case 5. An initial mixed entangled state (29) goes into a mixed entangled state with more or less entanglement depending on the choice of the parameter α:

$$\begin{aligned} \varrho &= \frac{3}{10}|2\rangle\langle 2| + \frac{1}{10}|3\rangle\langle 3| + \frac{3}{5}|4\rangle\langle 4| \\ \varrho_\infty &= \frac{2(1-\alpha)^2}{5(3+\alpha^2)}|1\rangle\langle 1| + \frac{2(1+\alpha)^2}{5(3+\alpha^2)}|2\rangle\langle 2| + \frac{2(1-\alpha^2)}{5(3+\alpha^2)}|3\rangle\langle 3| + \frac{3}{5}|4\rangle\langle 4| \qquad (29) \\ C(\varrho) &= \frac{1}{2}, \qquad C(\varrho_\infty) = \frac{3(1+3\alpha^2)}{5(3+\alpha^2)}. \end{aligned}$$

The concurrence of the asymptotic state $C(\varrho_\infty)$ can be larger or smaller than $C(\varrho) = \frac{1}{2}$ depending on the value of the parameter α (see figure 10).

If, for instance, we take $\alpha = 0.5$, then $C(\varrho_\infty) < \frac{1}{2}$, i.e. the asymptotic state has less entanglement than the initial state, and from the plot of the entanglement rate versus the entropy rate (figure 11) we can see that the conjecture (20) is always violated.

If, instead, we take for example $\alpha = 0.8$, then the initial entanglement first diminishes and then increases again, leading to an asymptotic state with more entanglement than the initial one, as can be seen from the plot of the entropy of entanglement as a function of time (figure 12). From the corresponding plot of the entanglement rate versus the entropy rate (figure 13), we can see that in this case the conjecture (20) is violated after some time.

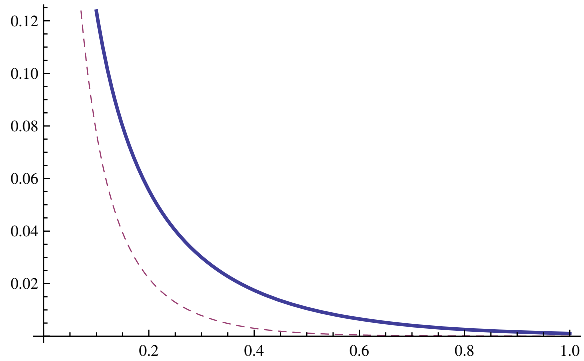

Figure 11. Case 5: $\alpha = 0.5$; $\sigma[\varrho_t]$ dashed line, $|\sigma_E[\varrho_t]|$ continuous line.

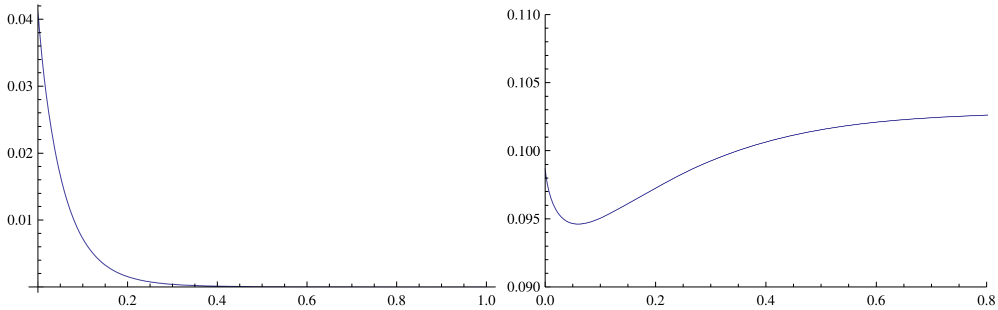

Figure 12. Case 5: $\alpha = 0.8$; left: $S(\varrho_t \| \varrho_\infty)$, right: $E[\varrho_t]$.

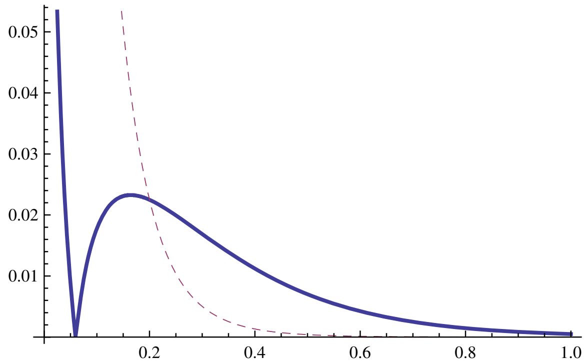

Figure 13. Case 5: $\alpha = 0.8$; $\sigma[\varrho_t]$ dashed line, $|\sigma_E[\varrho_t]|$ continuous line.

Remark 4. In the last plot of the entanglement rate versus the entropy rate, the cusp is due to the change of sign in the entropy of entanglement $S(\varrho_t \| \varrho_\infty)$ and to the fact that in the conjecture (20) the absolute value of the entanglement rate $\sigma_E[\varrho_t]$ is considered. On the other hand, all the other plots of the entanglement rate present a continuous behavior which reflects the fact that the entropy of entanglement does not change sign, i.e. it either increases or decreases monotonically. Finally, the plots of the relative entropy show its monotonic behavior under the action of completely positive trace-preserving maps; and to this corresponds a monotonic decreasing behavior for the entropy rate.

5. Conclusions

The time derivative of the quantum relative entropy serves as a measure of how fast an open quantum system tends to equilibrium dissipating free energy under a quantum dynamical semigroup of completely positive maps generated by a Lindblad-type master equation. On the other hand, via a variational formulation, the relative entropy may be used as a pseudo-distance of an entangled bipartite state from the convex subset of separable states (relative entropy of entanglement); therefore, its time derivative can be interpreted as the speed with which a time-evolving state moves toward, or away from, becoming separable.

Based on the expectation that the entanglement content of dissipatively driven bipartite systems disappears asymptotically due to decoherence effects, in [10] a conjecture was put forward, namely that the entropy rate, measured by (minus) the time derivative of the relative entropy of a dissipatively evolving state and its asymptotic state, should always be larger than the absolute value of the time derivative of the relative entropy of entanglement.

However, besides being a source of decoherence, an environment can in some cases build quantum correlations that can even persist asymptotically; in this paper we have studied the fate of the above conjecture in the case of a Lindblad-type master equation that presents a rich manifold of asymptotic states that may be more or less entangled with respect to the initial states they emerge from. The entropy and entanglement rates have been explicitly calculated and numerically plotted for a class of initial states. It turns out that, when the asymptotic state is entangled, the conjecture is violated either at all times or after a finite time; instead, the conjecture is confirmed in all cases when the asymptotic state is separable. The conjecture put forward in [10] should be thus reformulated as follows:

$$|\sigma_E[\varrho_t]| \leqslant \sigma[\rho_t] \quad \forall \rho_t \text{ with } \varrho_\infty \text{ separable.} \tag{30}$$

While the asymptotic predominance of the entropy rate over the entanglement rate in the latter case has already been explained in [10] based on the fact that there are no entangled states in a suitable neighborhood of the separable asymptotic state, the truly remarkable fact about (30) is its validity at all $t \geqslant 0$ in all the cases that have been checked.

Appendix A

In order to explicitly solve the master equation (A.1), one first writes the matrix A in diagonal form:

$$A = U \begin{pmatrix} 1+\alpha & 0 & 0 \\ 0 & 1-\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} U^\dagger, \qquad U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -\mathrm{i}/\sqrt{2} & \mathrm{i}/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and recasts the dissipative term in (3) in the form

$$\begin{aligned} \mathbf{D}[\varrho_t] = {} & 2(1+\alpha)\big(\Sigma_-\varrho_t\Sigma_+ - \tfrac{1}{2}\{\Sigma_+\Sigma_-, \varrho_t\}\big) + 2(1-\alpha)\big(\Sigma_+\varrho_t\Sigma_- - \tfrac{1}{2}\{\Sigma_-\Sigma_+, \varrho_t\}\big) \\ & + \Sigma_3\varrho_t\Sigma_3 - \tfrac{1}{2}\{\Sigma_3\Sigma_3, \varrho_t\}, \end{aligned} \tag{A.1}$$

with $\Sigma_\pm := \frac{1}{2}(\Sigma_1 \pm \mathrm{i}\Sigma_2)$. Since $H = \frac{\Omega}{2}\,\Sigma_3$, it follows that $\mathrm{e}^{\mathrm{i}Ht}\,\Sigma_\pm\,\mathrm{e}^{-\mathrm{i}Ht} = \mathrm{e}^{\pm\mathrm{i}\Omega t}\,\Sigma_\pm$. Thus, setting $\widetilde{\varrho}_t := \mathrm{e}^{\mathrm{i}Ht}\varrho_t\,\mathrm{e}^{-\mathrm{i}Ht}$ in the interaction picture, the master equation (A.1) becomes

$$\begin{aligned}\partial_t\widetilde{\varrho}_t &= \mathrm{e}^{\mathrm{i}Ht}\mathbf{D}[\mathrm{e}^{-\mathrm{i}Ht}\widetilde{\varrho}_t\mathrm{e}^{\mathrm{i}Ht}]\,\mathrm{e}^{-\mathrm{i}Ht}\\ &= 2(1+\alpha)\big(\Sigma_-\tilde{\varrho}_t\Sigma_+ - \tfrac{1}{2}\{\Sigma_+\Sigma_-,\tilde{\varrho}_t\}\big) + 2(1-\alpha)\big(\Sigma_+\tilde{\varrho}_t\Sigma_- - \tfrac{1}{2}\{\Sigma_-\Sigma_+,\tilde{\varrho}_t\}\big)\\ &\quad + \Sigma_3\tilde{\varrho}_t\Sigma_3 - \tfrac{1}{2}\{\Sigma_3\Sigma_3,\tilde{\varrho}_t\}.\end{aligned}\tag{A.2}$$

In order to solve this, it proves convenient to represent $\widetilde{\rho}_t = \sum_{i,j=1}^4 \rho_{ij}(t)|i\rangle\langle j|$ with respect to the orthonormal basis (6). Indeed, using that

$$\begin{cases}\Sigma_+|1\rangle = 0\\ \Sigma_+|2\rangle = \sqrt{2}|3\rangle\\ \Sigma_+|3\rangle = \sqrt{2}|1\rangle\\ \Sigma_+|4\rangle = 0,\end{cases}\qquad \begin{cases}\Sigma_-|1\rangle = \sqrt{2}|3\rangle\\ \Sigma_-|2\rangle = 0\\ \Sigma_-|3\rangle = \sqrt{2}|2\rangle\\ \Sigma_-|4\rangle = 0,\end{cases}\qquad \begin{cases}\Sigma_3|1\rangle = 2|1\rangle\\ \Sigma_3|2\rangle = -2|2\rangle\\ \Sigma_3|3\rangle = 0\\ \Sigma_3|4\rangle = 0,\end{cases}\tag{A.3}$$

one derives from (A.2) the following equations:

$$\begin{aligned}&\dot{\rho}_{11} = -4(1+\alpha)\rho_{11} + 4(1-\alpha)\rho_{33}, && \dot{\rho}_{12} = -12\rho_{12}\\ &\dot{\rho}_{13} = -2(4+\alpha)\rho_{13} + 4(1-\alpha)\rho_{32}, && \dot{\rho}_{14} = -2(2+\alpha)\rho_{14}\\ &\dot{\rho}_{22} = -4(1-\alpha)\rho_{22} + 4(1+\alpha)\rho_{33}, && \dot{\rho}_{23} = -2(4-\alpha)\rho_{23} + 4(1+\alpha)\rho_{31}\\ &\dot{\rho}_{33} = 4(1+\alpha)\rho_{11} + 4(1-\alpha)\rho_{22} - 8\rho_{33}, && \dot{\rho}_{24} = -2(2-\alpha)\rho_{24}\\ &\dot{\rho}_{34} = -4\rho_{34}, && \dot{\rho}_{44} = 0,\end{aligned}$$

plus the complex conjugated equations for $\dot{\rho}_{ij}$, $i \neq j$; whence

$$\begin{aligned}&\rho_{12}(t) = \rho_{12}\,\mathrm{e}^{-12t}, && \rho_{14}(t) = \rho_{14}\,\mathrm{e}^{-2(2+\alpha)t}\\ &\rho_{24}(t) = \rho_{24}\,\mathrm{e}^{-2(2-\alpha)t}, && \rho_{34}(t) = \rho_{34}\,\mathrm{e}^{-4t}\\ &\rho_{44}(t) = \rho_{44}.\end{aligned}\tag{A.4}$$

Of the remaining equations, two of them couple the off-diagonal terms ρ_{13} and ρ_{32}, yielding

$$\rho_{13}(t) = \rho_{13}F_+(t) + \frac{2(1-\alpha)\rho_{32} - \alpha\rho_{13}}{\sqrt{4-3\alpha^2}}F_-(t)\tag{A.5}$$

$$\rho_{32}(t) = \rho_{32}F_+(t) + \frac{2(1+\alpha)\rho_{13} + \alpha\rho_{32}}{\sqrt{4-3\alpha^2}}F_-(t),\tag{A.6}$$

while the other three solutions couple the diagonal entries:

$$\begin{aligned}\rho_{11}(t) = \frac{(1-\alpha)^2}{3+\alpha^2}R &+ \sqrt{1-\alpha^2}\,\frac{(1+\alpha)^2\rho_{11} - 2(1-\alpha)\rho_{22} + (1+\alpha)^2\rho_{33}}{(1+\alpha)(3+\alpha^2)}E_-(t)\\ &+ \frac{2(1+\alpha)\rho_{11} - (1-\alpha)^2(\rho_{22}+\rho_{33})}{3+\alpha^2}E_+(t)\end{aligned}\tag{A.7}$$

$$\begin{aligned}\rho_{22}(t) = \frac{(1+\alpha)^2}{3+\alpha^2}R &- \sqrt{1-\alpha^2}\,\frac{2(1+\alpha)\rho_{11} - (1-\alpha)^2(\rho_{22}+\rho_{33})}{(1-\alpha)(3+\alpha^2)}E_-(t)\\ &- \frac{(1+\alpha)^2\rho_{11} - 2(1+\alpha)\rho_{22} + (1+\alpha)^2\rho_{33}}{3+\alpha^2}E_+(t)\end{aligned}\tag{A.8}$$

$$\begin{aligned}\rho_{33}(t) = \frac{(1-\alpha^2)}{3+\alpha^2}R &+ \sqrt{1-\alpha^2}\,\frac{(1+\alpha)^3\rho_{11} + (1-\alpha)^3\rho_{22} - 2(1-\alpha^2)\rho_{33}}{(3+\alpha^2)(1-\alpha^2)}E_-(t)\\ &+ \frac{2(1+\alpha^2)\rho_{33} - (1-\alpha^2)(\rho_{11}+\rho_{22})}{3+\alpha^2}E_+(t),\end{aligned}\tag{A.9}$$

where $R = \rho_{11} + \rho_{22} + \rho_{33} = \rho_{11}(t) + \rho_{22}(t) + \rho_{33}(t)$ is a constant of the motion and

$$E_\pm(t) = \mathrm{e}^{-8t}\begin{cases}\cosh 4t\sqrt{1-\alpha^2}\\ \sinh 4t\sqrt{1-\alpha^2},\end{cases} \qquad F_\pm(t) = \mathrm{e}^{-8t}\begin{cases}\cosh 2t\sqrt{4-3\alpha^2}\\ \sinh 2t\sqrt{4-3\alpha^2}\end{cases} \tag{A.10}$$

are quantities which decay asymptotically with $t \to +\infty$. The remaining entries $\rho_{ij}(t)$ follow from complex conjugation. By returning to the Schrödinger representation, using (A.3), the explicit solution of (A.1) reads

$$\varrho_t = \sum_{i,j=1}^{4} \varrho_{ij}(t)\, \mathrm{e}^{2\mathrm{i}\omega t(\delta_{j1}+\delta_{i2}-\delta_{i1}-\delta_{j2})}\, |i\rangle\langle j|. \tag{A.11}$$

Appendix B

In order to prove the proposition in section 3, let us consider the spectral decompositions $\varrho_t = \sum_{i=1}^4 r_i(t)|i\rangle\langle i|$ (see (8)) and $\varrho_{\mathrm{sep}} = \sum_{j=1}^4 s_j|s_j\rangle\langle s_j|$. We have

$$\begin{aligned}\mathrm{Tr}(\varrho_t \log \varrho_{\mathrm{sep}}) &= \sum_{i=1}^4 r_i(t)\langle i|\log \varrho_{\mathrm{sep}}|i\rangle = \sum_{i=1}^4 r_i(t)\sum_{j=1}^4 |\langle i|s_j\rangle|^2 \log s_j\\ &\leqslant \sum_{i=1}^4 r_i(t)\log\left(\sum_{j=1}^4 s_j|\langle i|s_j\rangle|^2\right)\\ &= \sum_{i=1}^4 r_i(t)\log\langle i|\varrho_{\mathrm{sep}}|i\rangle = \mathrm{Tr}(\varrho_t \log \Pi[\varrho_{\mathrm{sep}}]),\end{aligned} \tag{B.1}$$

where the inequality follows from the convexity of $\log x$,

$$\log \sum_i \lambda_i x_i \geqslant \sum_i \lambda_i \log x_i, \quad \lambda_i \geqslant 0, \qquad \sum_i \lambda_i = 1, \quad x_i \geqslant 0,$$

and $\sum_{j=1}^4 |\langle i|s_j\rangle|^2 = 1$. Also, we have introduced the completely positive map

$$\varrho \mapsto \Pi[\varrho] := \sum_{i=1}^4 |i\rangle\langle i|\varrho|i\rangle\langle i|, \tag{B.2}$$

on the 2-qubit density matrices $\mathcal{S}(\mathbb{C}^4)$ that diagonalize its argument with respect to the orthonormal basis (6). This map has the following property which allows one to analytically solve the variational problem (B.1).

Lemma. $\Pi : \mathcal{S}(\mathbb{C}^4) \mapsto \mathcal{S}(\mathbb{C}^4)$ *maps separable states into separable states.*

Proof. Given the density matrix of an arbitrary 2-qubit state in the standard basis,

$$\varrho = \begin{pmatrix}\varrho_{00,00} & \varrho_{00,01} & \varrho_{00,10} & \varrho_{00,11}\\ \varrho_{01,00} & \varrho_{01,01} & \varrho_{01,10} & \varrho_{01,11}\\ \varrho_{10,00} & \varrho_{10,01} & \varrho_{10,10} & \varrho_{10,11}\\ \varrho_{11,00} & \varrho_{11,01} & \varrho_{11,10} & \varrho_{11,11}\end{pmatrix},$$

the action of the map Π transforms it into a density matrix of the form

$$\Pi[\varrho] = \begin{pmatrix}\varrho_{00,00} & 0 & 0 & 0\\ 0 & \frac{\varrho_{01,01}+\varrho_{10,10}}{2} & \Re\mathrm{e}(\varrho_{01,10}) & 0\\ 0 & \Re\mathrm{e}(\varrho_{01,10}) & \frac{\varrho_{01,01}+\varrho_{10,10}}{2} & 0\\ 0 & 0 & 0 & \varrho_{11,11}\end{pmatrix}.$$

By partial transposition [20], $\Pi[\varrho]$ is entangled if and only if $|\Re\mathrm{e}(\varrho_{01,10})| \geqslant \sqrt{\varrho_{00,00}\varrho_{11,11}}$. But then, the partially transposed ϱ (with respect to the second qubit),

$$\varrho^\Gamma = \begin{pmatrix} \varrho_{00,00} & \varrho_{01,00} & \varrho_{00,10} & \varrho_{01,10} \\ \varrho_{00,01} & \varrho_{01,01} & \varrho_{00,11} & \varrho_{01,11} \\ \varrho_{10,00} & \varrho_{11,00} & \varrho_{10,10} & \varrho_{11,10} \\ \varrho_{10,01} & \varrho_{11,01} & \varrho_{10,11} & \varrho_{11,11} \end{pmatrix},$$

cannot be positive semi-definite, for $|\varrho_{01,10}| \geqslant |\Re\mathrm{e}(\varrho_{01,10})| > \sqrt{\varrho_{00,00}\varrho_{11,11}}$ in the sub-matrix $\left(\begin{smallmatrix} \varrho_{11} & \varrho_{01,10} \\ \varrho_{10,01} & \varrho_{22} \end{smallmatrix}\right)$. Therefore, if ϱ is separable, then also $\Pi[\varrho]$ must be so. □

Observe that (B.1) implies $\sup_{\varrho_{\rm sep}} \mathrm{Tr}(\varrho_t \log \varrho_{\rm sep}) \leqslant \sup_{\varrho_{\rm sep}} \mathrm{Tr}(\varrho_t \log \Pi[\varrho_{\rm sep}])$; on the other hand, since Π maps separable states into separable states,

$$\sup_{\varrho_{\rm sep}} \mathrm{Tr}(\varrho_t \log \varrho_{\rm sep}) \leqslant \sup_{\varrho_{\rm sep}} \mathrm{Tr}(\varrho_t \log \Pi[\varrho_{\rm sep}]) \leqslant \sup_{\varrho_{\rm sep}} \mathrm{Tr}(\varrho_t \log \varrho_{\rm sep}).$$

Thus, the maximum in (B.1) is attained on the subset $\mathcal{S}_{\rm sep}^{\rm diag}$ of separable qubit states that are diagonal with respect to the orthonormal basis, namely of the form (22) with the second bound on the real parameters x, y, u, v in (23) coming from the condition of positivity under partial transposition of matrices of the form (7), which is necessary and sufficient for separability.

It thus follows from (21) and (B.1) that for ϱ_t as in (8) the relative entropy of entanglement can be reduced to the computation of (24). In order to explicitly solve such a variational problem, we seek the stationary points of a function of the form

$$f(x, y, u, v) := a \log x + d \log y + b \log u + c \log v + \lambda(x + y + u + v - 1),$$

with given $a, b, c, d \geqslant 0$ such that $a + b + c + d = 1$, relative to variations of the parameters x, y, u, v over values achieving separable states of the form (22). Stationarity implies

$$a = -\lambda x, \qquad d = -\lambda y, \qquad b = -\lambda u, \qquad c = -\lambda v;$$

whence $\lambda = -1$ and $a = x, d = y, b = u, c = v$. However, this can be the required solution only if the state ϱ in $E[\varrho]$ is separable so that $E[\varrho] = 0$. Otherwise, the solution must lie on the border of the subset of separable states of the form (22), where the inequality in (23) is saturated. From $x + u + v + y = 1$ and $(u - v)^2 = 4\,x\,y$, one gets

$$u_\pm = \frac{1 - (\sqrt{x} \mp \sqrt{y})^2}{2}, \qquad v_\pm = \frac{1 - (\sqrt{x} \pm \sqrt{y})^2}{2},$$

so that the function to be maximized becomes

$$f(x, y) = a \log x + d \log y + b \log\left(\frac{1 - (\sqrt{x} \mp \sqrt{y})^2}{2}\right) + c \log\left(\frac{1 - (\sqrt{x} \pm \sqrt{y})^2}{2}\right). \tag{B.3}$$

Stationarity with respect to x, y leads to a system of two equations for the two unknowns x and y in terms of the coefficients a, b, c, d. From setting $\partial_{x,y} f(x, y) = 0$ and from the condition $a + b + c + d = 1$ it follows that $y = x - a + d$ and that

$$\begin{aligned} x = \frac{1}{8(-1+b)(a+b+d)} \{ & -a^3 - d(-1+2b+d)^2 + a^2(-6+4b+d) \\ & + a(-1+4(-1+b)+d^2) + [(-1+a+2b+d)^2(a^4+2a^3(-1+2b-2d) \\ & + d^2(-1+2b+d)^2 + a^2(1+4b^2+2d+6d^2-4b(1+d)) \\ & + 2ad(1+d-2(2b(-1+b)+bd+d^2)))]^{1/2}\}. \end{aligned}$$

By inserting into it the values (9)–(11), this expression yields the separable state of the form (22) which is closest to an evolving entangled state of the form (8). Though cumbersome, the resulting entanglement rate (19) is amenable to numerical inspection.

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